A KRASNOSELSKII-TYPE FIXED POINT THEOREM FOR MULTIFUNCTIONS DEFINED ON A HYPERCONVEX SPACE

MARCIN BORKOWSKI

Abstract. We present a fixed point theorem for a sum of two convex-valued multifunctions acting on a weakly compact, hyperconvex subset of a normed space. The theorem is a multivalued version of a result of D. Bugajewski.

1. Introduction

In [1, s. 1458, Theorem 2], D. Bugajewski proved the following Krasnoselskii-type theorem in a hyperconvex setting.

Theorem 1. *Let K be a bounded hyperconvex subset of a normed space $(X, \|\cdot\|)$ such that $\lambda K \subset K$ for every $\lambda \in (0, 1]$. Assume that*

(1) *$f_1 \colon K \to X$ is nonexpansive;*
(2) *$f_2 \colon K \to X$ is completely continuous;*
(3) *$f_1(x) + f_2(y) \in K$ for any $x, y \in K$;*
(4) *every sequence (x_n) such that $x_n \in K$ for $n \in \mathbb{N}$ and*
$$\lim_{n\to\infty} \bigl(x_n - f(x_n)\bigr) = 0,$$
where $f := f_1 + f_2$, has a limit point.

Then, f has a fixed point.

Recall that the assumption that $\lambda H \subset H$ can be released, as it was shown in [2].

Recently, M. Özdemir and S. Akbulut published the paper [3] with a multivalued version of Bugajewski's theorem. Unfortunately, their proof contains some errors. We will state a slightly different version of this theorem and then discuss the errors in [3].

2. Preliminaries

Let X be a metric space. By $\operatorname{Bd} X$ we denote the family of nonempty, bounded and closed subsets of X and by $\operatorname{Komp} X$ the set of nonempty compact subsets of X.

In what follows, we will use the symbol d_X for a metric in the space X and H_X for the Hausdorff metric in the hyperspace $\operatorname{Bd} X$; we will write d and H if the underlying space is obvious from the context.

By A' we will denote the complement of the subset A of some space X, i.e., the set $X \setminus A$.

Definition 1. We call a mapping $f\colon X \to Y$ between metric spaces *nonexpansive*, if $d(f(x), f(y)) \le d(x, y)$ for each $x, y \in X$.

Definition 2. Let A be any subset of a metric space X. The *Kuratowski measure of noncompactness* of the set A, denoted by $\alpha(A)$, is the greatest lower bound of the numbers $\varepsilon > 0$ such that A can be covered by a finite family of sets of diameter not greater than ε. (We put $\alpha(A) = +\infty$ for unbounded sets.) A mapping $f\colon X \to Y$ between metric spaces is called *α-condensing* if $\alpha(f(A)) \le \alpha(A)$ for each nonempty $A \subset X$ and $\alpha(f(A)) < \alpha(A)$ provided that $\alpha(A) > 0$.

References

[1] D. Bugajewski, *Fixed-point theorems in hyperconvex spaces revisited*, Math. Comput. Modelling **32** (2000), no. 13, 1457–1461.

[2] D. Bugajewski and R. Espínola, *Remarks on some fixed point theorems for hyperconvex spaces and absolute retracts*, Function Spaces the Fifth Conference (Lecture Notes in Pure and Applied Mathematics Series 213). Edited by H. Hudzik and L. Skrzypczak, 2000, pp. 85–92.

[3] M. Özdemir and S. Akbulut, *A fixed point theorem for multivalued maps in hyperconvex spaces*, Appl. Math. Comput. **157** (2004), 637–642.

Optimization and Control Theory Department, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland

E-mail address: mbork@amu.edu.pl